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Commutators of integral operators with a function in generalized Campanato spaces

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Dedicated to the memory of Professor Yasuji Takahashi

1 Introduction

This is an announcement of [1].

Let $\mathbb{R}^n$ be the n-dimensional Euclidean space. Let $b \in \mathrm{BMO}(\mathbb{R}^n)$ and T be a Calderón-Zygmund singular integral operator. In 1976 Coifman, Rochberg and Weiss [3] proved that the commutator $[b, T] = bT - Tb$ is bounded on $L^p(\mathbb{R}^n)$ $(1 < p < \infty)$, that is,

$$\|[b,T]f\|_{L^p} = \|bTf - T(bf)\|_{L^p} \le C\|b\|_{\mathrm{BMO}}\|f\|_{L^p},$$

where C is a positive constant independent of b and f. For the fractional integral operator I_α, Chanillo [2] proved the boundedness of $[b, I_\alpha]$ in 1982. That is,

$$\|[b,I_\alpha]f\|_{L^q} \le C\|b\|_{\mathrm{BMO}}\|f\|_{L^p},$$

where $\alpha \in (0,n)$, $p,q \in (1,\infty)$ and $-n/p + \alpha = -n/q$. These results were extended to Morrey spaces by Di Fazio and Ragusa [4] in 1991.

In this talk we discuss the boundedness of the commutators $[b, T]$ and $[b, I_\rho]$ on generalized Morrey spaces with variable growth condition, where T is a Calderón-Zygmund operator, I_ρ is a generalized fractional integral operator and b is a function in generalized Campanato spaces with variable growth condition.

We denote by $B(x,r)$ the open ball centered at $x \in \mathbb{R}^n$ and of radius r, that is,

$$B(x,r) = \{y \in \mathbb{R}^n : |y - x| < r\}.$$

For a measurable set $G \subset \mathbb{R}^n$, we denote by $|G|$ and χ_G the Lebesgue measure of G and the characteristic function of G, respectively. For a function $f \in L^1_{\mathrm{loc}}(\mathbb{R}^n)$ and a ball B, let

$$f_B = \fint_B f = \fint_B f(y)\,dy = \frac{1}{|B|}\int_B f(y)\,dy.$$

For a variable growth function $\varphi : \mathbb{R}^n \times (0,\infty) \to (0,\infty)$ and a ball $B = B(x,r)$ we write $\varphi(B) = \varphi(x,r)$.

Definition 1.1. For $p \in [1,\infty)$ and $\varphi : \mathbb{R}^n \times (0,\infty) \to (0,\infty)$, let $L^{(p,\varphi)}(\mathbb{R}^n)$ be the sets of all functions f such that the following functional is finite:

$$\|f\|_{L^{(p,\varphi)}(\mathbb{R}^n)} = \sup_B \left(\frac{1}{\varphi(B)} \fint_B |f(y)|^p \, dy \right)^{1/p},$$

where the supremum is taken over all balls B in $\mathbb{R}^n$.

Then $\|f\|_{L^{(p,\varphi)}(\mathbb{R}^n)}$ is a norm and $L^{(p,\varphi)}(\mathbb{R}^n)$ is a Banach space. If $\varphi_\lambda(x,r) = r^\lambda$ with $\lambda \in [-n, 0]$, then $L^{(p,\varphi_\lambda)}(\mathbb{R}^n)$ is the classical Morrey spaces. If $\lambda = -n$, then $L^{(p,\varphi_{-n})}(\mathbb{R}^n) = L^p(\mathbb{R}^n)$. If $\lambda = 0$, then $L^{(p,\varphi_0)}(\mathbb{R}^n) = L^\infty(\mathbb{R}^n)$.

Definition 1.2. For $p \in [1,\infty)$ and $\varphi : \mathbb{R}^n \times (0,\infty) \to (0,\infty)$, let $\mathcal{L}^{(p,\varphi)}(\mathbb{R}^n)$ be the sets of all functions f such that the following functional is finite:

$$\|f\|_{\mathcal{L}^{(p,\varphi)}(\mathbb{R}^n)} = \sup_B \left(\frac{1}{\varphi(B)} \fint_B |f(y) - f_B|^p \, dy \right)^{1/p},$$

where the supremum is taken over all balls B in $\mathbb{R}^n$.

Then $\|f\|_{\mathcal{L}^{(p,\varphi)}(\mathbb{R}^n)}$ is a norm modulo constant functions and thereby $\mathcal{L}^{(p,\varphi)}(\mathbb{R}^n)$ is a Banach space. If $p = 1$ and $\varphi \equiv 1$, then $\mathcal{L}^{(p,\varphi)}(\mathbb{R}^n) = \mathrm{BMO}(\mathbb{R}^n)$. If $p = 1$ and $\varphi(r) = r^\alpha$ $(0 < \alpha \le 1)$, then $\mathcal{L}^{(p,\varphi)}(\mathbb{R}^n) = \mathrm{Lip}_\alpha(\mathbb{R}^n)$.

A linear operator T from $\mathcal{S}(\mathbb{R}^n)$ to $\mathcal{S}'(\mathbb{R}^n)$ is said to be a Calderón-Zygmund operator if T is bounded on $L^2(\mathbb{R}^n)$ and there exists a standard kernel K such that, for $f \in C^\infty_{\mathrm{comp}}(\mathbb{R}^n)$,

$$Tf(x) = \int_{\mathbb{R}^n} K(x,y) f(y) \, dy, \quad x \notin \operatorname{supp} f.$$

It is known that any Calderón-Zygmund operator T is bounded on $L^p(\mathbb{R}^n)$ for $1 < p < \infty$.

For a function $\rho : \mathbb{R}^n \times (0,\infty) \to (0,\infty)$, we consider generalized fractional integral operators I_ρ defined by

$$I_\rho f(x) = \int_{\mathbb{R}^n} \frac{\rho(x, |x-y|)}{|x-y|^n} f(y) \, dy,$$

where we always assume that

$$\int_0^1 \frac{\rho(x,t)}{t} \, dt < \infty \quad \text{for each } x \in \mathbb{R}^n. \tag{1.1}$$

and that there exist positive constants C, K_1 and K_2 with $K_1 < K_2$ such that

$$\sup_{r \le t \le 2r} \rho(x,t) \le C \int_{K_1 r}^{K_2 r} \frac{\rho(x,t)}{t} \, dt \quad \text{for all } x \in \mathbb{R}^n \text{ and } r > 0. \tag{1.2}$$

If $\rho(x,r) = r^\alpha$, then I_ρ is the usual fractional integral operator I_α. It is known as the Hardy-Littlewood-Sobolev theorem that I_α is bounded from $L^p(\mathbb{R}^n)$ to $L^q(\mathbb{R}^n)$, if $\alpha \in (0,n)$, $p, q \in (1,\infty)$ and $-n/p + \alpha = -n/q$.

2 Main results

We say that θ is almost increasing (resp. almost decreasing) if there exists a positive constant C such that, for all $x \in \mathbb{R}^n$ and $r, s \in (0, \infty)$,

$$\theta(x, r) \le C\theta(x, s) \quad (\text{resp. } \theta(x, s) \le C\theta(x, r)), \quad \text{if } r < s.$$

In this talk we consider the following classes of φ:

Definition 2.1. (i) Let $\mathcal{G}^{dec}$ be the set of all functions $\varphi : \mathbb{R}^n \times (0, \infty) \to (0, \infty)$ such that φ is almost decreasing, and that $r \mapsto \varphi(x, r)r^n$ is almost increasing. (ii) Let $\mathcal{G}^{inc}$ be the set of all functions $\varphi : \mathbb{R}^n \times (0, \infty) \to (0, \infty)$ such that φ is almost increasing, and that $r \mapsto \varphi(x, r)/r$ is almost decreasing.

Let $\varphi \in \mathcal{G}^{dec}$. If φ satisfies

$$\lim_{r\to 0} \varphi(x, r) = \infty, \quad \lim_{r\to\infty} \varphi(x, r) = 0, \tag{2.1}$$

then there exists $\tilde{\varphi} \in \mathcal{G}^{dec}$ such that $\varphi \sim \tilde{\varphi}$ and that $\varphi(x, \cdot)$ is continuous, strictly decreasing and bijective from $(0, \infty)$ to itself for each x.

We also consider the following conditions:

$\exists C > 0\ \forall x, y \in \mathbb{R}^n\ \forall r \in (0, \infty)$,

$$\frac{1}{C} \le \frac{\theta(x, r)}{\theta(y, r)} \le C, \quad \text{if } |x - y| \le r. \tag{2.2}$$

$\exists C > 0\ \forall x \in \mathbb{R}^n\ \forall r \in (0, \infty)$,

$$\int_r^\infty \frac{\varphi(x, t)}{t}\, dt \le C\varphi(x, r). \tag{2.3}$$

For functions f in Morrey spaces, we define $[b, T]f$ on each ball B by

$$[b, T]f(x) = [b, T](f\chi_{2B})(x) + \int_{\mathbb{R}^n\setminus 2B} (b(x) - b(y))K(x, y)f(y)\, dy, \quad x \in B.$$

Then we have the following theorem.

Theorem 2.1. *Let $1 < p \le q < \infty$ and $\varphi, \psi : \mathbb{R}^n \times (0, \infty) \to (0, \infty)$. Assume that $\varphi \in \mathcal{G}^{dec}$ and $\psi \in \mathcal{G}^{inc}$. Let T be a Calderón-Zygmund operator.*

(i) *Assume that ψ satisfy (2.2), that φ satisfies (2.3), and that there exists a positive constant C_0 such that, for all $x \in \mathbb{R}^n$ and $r \in (0, \infty)$,*

$$\psi(x, r)\varphi(x, r)^{1/p} \le C_0\varphi(x, r)^{1/q}.$$

If $b \in \mathcal{L}^{(1,\psi)}(\mathbb{R}^n)$, then $[b, T]f$ is well defined for all $f \in L^{(p,\varphi)}(\mathbb{R}^n)$ and there exists a positive constant C, independent of b and f, such that

$$\|[b, T]f\|_{L^{(q,\varphi)}} \le C\|b\|_{\mathcal{L}^{(1,\psi)}}\|f\|_{L^{(p,\varphi)}}.$$

(ii) *Conversely, assume that* φ *satisfies* (2.2) *and that there exists a positive constant* C_0 *such that, for all* $x \in \mathbb{R}^n$ *and* $r \in (0,\infty)$,

$$C_0\psi(x,r)\varphi(x,r)^{1/p} \geq \varphi(x,r)^{1/q}.$$

If T *is a convolusion type such that*

$$Tf(x) = p.v. \int_{\mathbb{R}^n} K(x-y)f(y)\,dy$$

with homogeneous kernel K *satisfying* $K(x) = |x|^{-n}K(x/|x|)$, $\int_{S^{n-1}} K = 0$ *and* $K \in C^\infty(S^{n-1})$ *and* $K \not\equiv 0$, *and if* $[b,T]$ *is bounded from* $L^{(p,\varphi)}(\mathbb{R}^n)$ *to* $L^{(q,\varphi)}(\mathbb{R}^n)$, *then* $b \in \mathcal{L}^{(1,\psi)}(\mathbb{R}^n)$ *and there exists a positive constant* C, *independent of* b, *such that*

$$\|b\|_{\mathcal{L}^{(1,\psi)}} \leq C\|[b,T]\|_{L^{(p,\varphi)}\to L^{(q,\varphi)}},$$

where $\|[b,T]\|_{L^{(p,\varphi)}\to L^{(q,\varphi)}}$ *is the opetator norm of* $[b,T]$ *from* $L^{(p,\varphi)}(\mathbb{R}^n)$ *to* $L^{(q,\varphi)}(\mathbb{R}^n)$.

In the above theorem, if $\psi \equiv 1$ and $\varphi(x,r) = r^{-n}$, then $\mathcal{L}^{(1,\psi)}(\mathbb{R}^n) = \mathrm{BMO}(\mathbb{R}^n)$ and $L^{(p,\varphi)}(\mathbb{R}^n) = L^p(\mathbb{R}^n)$. This is the case of the theorem by Coifman, Rochberg and Weiss.

If $\psi(x,r) = r^\alpha$, $0 < \alpha \leq 1$, and $\varphi(x,r) = r^{-n}$, then $\mathcal{L}^{(1,\psi)}(\mathbb{R}^n) = \mathrm{Lip}_\alpha(\mathbb{R}^n)$, $L^{(p,\varphi)}(\mathbb{R}^n) = L^p(\mathbb{R}^n)$ and $L^{(q,\varphi)}(\mathbb{R}^n) = L^q(\mathbb{R}^n)$ with $-n/p + \alpha = -n/q$. That is,

$$\|[b,T]f\|_{L^q} \lesssim \|b\|_{\mathrm{Lip}_\alpha}\|f\|_{L^p}.$$

This is the case of Janson [5, Lemma 12].

Theorem 2.2. *Let* $1 < p < q < \infty$ *and* $\rho, \varphi, \psi : \mathbb{R}^n \times (0,\infty) \to (0,\infty)$. *Assume that* $\varphi \in \mathcal{G}^{dec}$ *and* $\psi \in \mathcal{G}^{inc}$. *Assume also that* ρ *satisfies* (1.1) *and* (1.2). *Let* $\rho^*(x,r) = \int_0^r \frac{\rho(x,t)}{t}\,dt$.

(i) *Assume that* ρ, ρ^* *and* ψ *satisfy* (2.2), *that* φ *satisfies* (2.3) *and that there exist positive constants* ϵ, C_ρ, C_0, C_1 *and an exponent* $\tilde{p} \in (p,q]$ *such that, for all* $x,y \in \mathbb{R}^n$ *and* $r,s \in (0,\infty)$,

$$C_\rho \frac{\rho(x,r)}{r^{n-\epsilon}} \geq \frac{\rho(x,s)}{s^{n-\epsilon}}, \quad \text{if } r < s, \tag{2.4}$$

$$\left|\frac{\rho(x,r)}{r^n} - \frac{\rho(y,s)}{s^n}\right| \leq C_\rho\,(|r-s| + |x-y|)\frac{\rho^*(x,r)}{r^{n+1}}, \quad \text{if } \frac{1}{2} \leq \frac{r}{s} \leq 2 \text{ and } |x-y| < r/2, \tag{2.5}$$

$$\int_0^r \frac{\rho(x,t)}{t}\,dt\,\varphi(x,r)^{1/p} + \int_r^\infty \frac{\rho(x,t)\varphi(x,t)^{1/p}}{t}\,dt \leq C_0\varphi(x,t)^{1/\tilde{p}}, \tag{2.6}$$

$$\psi(x,r)\varphi(x,r)^{1/\tilde{p}} \leq C_1\varphi(x,r)^{1/q}. \tag{2.7}$$

If $b \in \mathcal{L}^{(1,\psi)}(\mathbb{R}^n)$, then $[b, I_\rho]f$ is well defined for all $f \in L^{(p,\varphi)}(\mathbb{R}^n)$ and there exists a positive constant C, independent of b and f, such that

$$\|[b, I_\rho]f\|_{L^{(q,\varphi)}} \leq C\|b\|_{\mathcal{L}^{(1,\psi)}}\|f\|_{L^{(p,\varphi)}}.$$

(ii) *Conversely, assume that φ satisfies (2.2), that $\rho(x, r) = r^\alpha$, $0 < \alpha < n$, and that*

$$C_0\psi(x, r)r^\alpha\, \varphi(x, r)^{1/p} \geq \varphi(x, r)^{1/q}. \tag{2.8}$$

If $[b, I_\alpha]$ is bounded from $L^{(p,\varphi)}(\mathbb{R}^n)$ to $L^{(q,\varphi)}(\mathbb{R}^n)$, then $b \in \mathcal{L}^{(1,\psi)}(\mathbb{R}^n)$ and there exists a positive constant C, independent of b, such that

$$\|b\|_{\mathcal{L}^{(1,\psi)}} \leq C\|[b, I_\alpha]\|_{L^{(p,\varphi)}\to L^{(q,\varphi)}},$$

where $\|[b, I_\alpha]\|_{L^{(p,\varphi)}\to L^{(q,\varphi)}}$ is the opetator norm of $[b, I_\alpha]$ from $L^{(p,\varphi)}(\mathbb{R}^n)$ to $L^{(q,\varphi)}(\mathbb{R}^n)$.

3 Sketch of proof

We give a sketch of the proof of Theorem 2.2. To prove the theorem we use the following inequality and theorem:

$$M^\sharp([b, I_\rho]f)(x) \;\leq\; C\|b\|_{\mathcal{L}^{(1,\psi)}}\Big(\big(M_{\psi^\eta}(|I_\rho f|^\eta)(x)\big)^{1/\eta} + \big(M_{(\rho^*\psi)^\eta}(|f|^\eta)(x)\big)^{1/\eta}\Big),$$

where $1 < \eta < \infty$, $\rho^*(x, r) = \int_0^r \rho(x, t)t^{-1}\, dt$ and

$$M^\sharp f(x) = \sup_{B\ni x} \fint_B |f(y) - f_B|\, dy, \quad M_\rho f(x) = \sup_{B\ni x} \rho(B) \fint_B |f(y)|\, dy.$$

Theorem 3.1 (Nakai, 2014). *Let $p \in [1, \infty)$ be a constant and $\varphi : \mathbb{R}^n \times (0, \infty) \to (0, \infty)$. Assume that there exists a positive constant C such that,*

$$\varphi(x, r) \geq C\varphi(x, s) \text{ for all } x \in \mathbb{R}^n \text{and } r \in (0, s).$$

Then the operator M is bounded from $L^{(p,\varphi)}(\mathbb{R}^n)$ to itself if $p \in (1, \infty)$.

Theorem 3.2 (Nakai, 2014). *Let $1 < p < q < \infty$ and $\rho, \varphi : \mathbb{R}^n \times (0, \infty) \to (0, \infty)$. Assume that ρ satisfies (1.1) and (1.2) and that φ is in $\mathcal{G}^{\mathrm{dec}}$ and satisfies (2.1). Assume also that there exists a positive constant C such that, for all $x \in \mathbb{R}^n$ and $r \in (0, \infty)$,*

$$\int_0^r \frac{\rho(x, t)}{t}\, dt\, \varphi(x, r)^{1/p} + \int_r^\infty \frac{\rho(x, t)\varphi(x, t)^{1/p}}{t}\, dt \leq C\varphi(x, r)^{1/q}.$$

Then I_ρ is bounded from $L^{(p,\varphi)}(\mathbb{R}^n)$ to $L^{(q,\varphi)}(\mathbb{R}^n)$.

Theorem 3.3. *Let $1 < p < q < \infty$ and $\rho, \varphi : \mathbb{R}^n \times (0,\infty) \to (0,\infty)$. Assume that φ is in $\mathcal{G}^{dec}$ and satisfies (2.1). Assume also that*

$$\rho(x,r)\varphi(x,r)^{1/p} \le C_0\varphi(x,r)^{1/q}. \tag{3.1}$$

Then M_ρ is bounded from $L^{(p,\varphi)}(\mathbb{R}^n)$ to $L^{(q,\varphi)}(\mathbb{R}^n)$.

Proof. We may assume that $\varphi(x,\cdot)$ is continuous, strictly decreasing and bijective from $(0,\infty)$ to itself for each $x \in \mathbb{R}^n$.

We prove that, for $f \in L^{(p,\varphi)}(\mathbb{R}^n)$ with $\|f\|_{L^{(p,\varphi)}(\mathbb{R}^n)} = 1$,

$$M_\rho f(x) \le CMf(x)^{p/q}, \quad x \in \mathbb{R}^n, \tag{3.2}$$

for some positive constant C independent of f and x. To prove (3.2) we show that, for any ball $B = B(x,r)$,

$$\rho(B) \fint_B |f| \le C_0 Mf(x)^{p/q}. \tag{3.3}$$

Choose $u > 0$ such that $\varphi(x,u) = Mf(x)^p$. If $r \le u$, then $\varphi(B) = \varphi(x,r) \ge Mf(x)^p$ and $\varphi(B)^{1/q-1/p} \le Mf(x)^{p/q-1}$. By (3.1) we have

$$\rho(B) \fint_B |f| \le C_0\varphi(B)^{1/q-1/p} \fint_B |f| \le C_0 Mf(x)^{p/q}.$$

If $r > u$, then $\varphi(B) = \varphi(x,r) < Mf(x)^p$ and $\varphi(B)^{1/q} < Mf(x)^{p/q}$. By (3.1) we have

$$\rho(B) \fint_B |f| \le \rho(B) \left(\fint_B |f|^p \right)^{1/p} \le \rho(B)\varphi(B)^{1/p} \le C_0\varphi(B)^{1/q} \le C_0 Mf(x)^{p/q}.$$

Then we have (3.3) and the conclusion. □

Proposition 3.4. *Let $1 \le p < \infty$ and $\varphi : \mathbb{R}^n \times (0,\infty) \to (0,\infty)$. Then, for $f \in L^1_{\mathrm{loc}}(\mathbb{R}^n)$,*

$$\|f\|_{\mathcal{L}^{(p,\varphi)}} \le C\|M^\sharp f\|_{L^{(p,\varphi)}}, \tag{3.4}$$

where C is a positive constant independent of f.

Corollary 3.5. *Let $1 \le p < \infty$ and $\varphi : \mathbb{R}^n \times (0,\infty) \to (0,\infty)$. Assume that $\varphi \in \mathcal{G}^{\mathrm{dec}}$ and that φ satisfies (2.3). For $f \in L^1_{\mathrm{loc}}(\mathbb{R}^n)$, if $\lim_{r\to\infty} f_{B(0,r)} = 0$, then*

$$\|f\|_{L^{(p,\varphi)}} \le C\|M^\sharp f\|_{L^{(p,\varphi)}}, \tag{3.5}$$

where C is a positive constant independent of f.

Lemma 3.6 ([8, Theorem 2.1 and Remark 2.1])**.** *Let $p \in [1,\infty)$ and φ is in $\mathcal{G}^{dec}$ and satisfies (2.3). Then, for every $f \in \mathcal{L}^{(p,\varphi)}(\mathbb{R}^n)$, $f_{B(0,r)}$ converges as $r \to \infty$ and*

$$\|f - \lim_{r\to\infty} f_{B(0,r)}\|_{L^{(p,\varphi)}} \sim \|f\|_{\mathcal{L}^{(p,\varphi)}},$$

For any cube $Q \subset \mathbb{R}^n$ centered at $a \in \mathbb{R}^n$ and with sidelength $2r > 0$, we denote by $\mathcal{Q}^{\mathrm{dy}}(Q)$ the set of all dyadic cubes with respect to Q.

For any cube $Q \subset \mathbb{R}^n$, let

$$M_Q^{\mathrm{dy}} f(x) = \sup_{R \in \mathcal{Q}^{\mathrm{dy}}(Q),\, x \in R \subset Q} \fint_Q |f(y)|\, dy,$$

$$M_Q^{\sharp,\mathrm{dy}} f(x) = \sup_{R \in \mathcal{Q}^{\mathrm{dy}}(Q),\, x \in R \subset Q} \fint_Q |f(y) - f_Q|\, dy.$$

Lemma 3.7 (Tsutsui, 2011 Komori, 2015)**.** *Let Q be a cube and $f \in L^1(Q)$. Then, for any $0 < \gamma \le 1$ and $\lambda > |f|_Q$,*

$$\begin{aligned} &|\{x \in Q :\, M_Q^{\mathrm{dy}} f(x) > 2\lambda, M_Q^{\sharp,\mathrm{dy}} f(x) \le \gamma\lambda\}| \\ &\qquad \le 2^n \gamma |\{x \in Q :\, M_Q^{\mathrm{dy}} f(x) > \lambda\}|. \end{aligned} \tag{3.6}$$

Lemma 3.8. *There exists a positive constant C, for any cube Q and any function $f \in L^1(Q)$,*

$$\|f - f_Q\|_{L^p(Q)} \le C \|M_Q^{\sharp,\mathrm{dy}} f\|_{L^p(Q)}.$$

Proof. By the good λ inequality (3.6) and the standard argument we have the following boundedness: There exists a positive constant C, for any cube Q and any function $f \in L^1(Q)$,

$$\|M_Q^{\mathrm{dy}} f\|_{L^p(Q)} \le C \left(\|M_Q^{\sharp,\mathrm{dy}} f\|_{L^p(Q)} + |Q|^{1/p} |f|_Q \right). \tag{3.7}$$

Actually, for any $L > 2|f|_Q$,

$$\begin{aligned} &\int_0^L p\lambda^{p-1} |\{x \in Q : M_Q^{\mathrm{dy}} f(x) > \lambda\}|\, d\lambda \\ &= \int_0^{2|f|_Q} p\lambda^{p-1} |\{x \in Q : M_Q^{\mathrm{dy}} f(x) > \lambda\}|\, d\lambda \\ &\qquad + \int_{2|f|_Q}^L p\lambda^{p-1} |\{x \in Q : M_Q^{\mathrm{dy}} f(x) > \lambda\}|\, d\lambda \\ &\le (2|f|_Q)^p |Q| + 2^p \int_{|f|_Q}^{L/2} p\lambda^{p-1} |\{x \in Q : M_Q^{\mathrm{dy}} f(x) > 2\lambda\}|\, d\lambda. \end{aligned}$$

By the good λ inequality (3.6) we have

$$
\begin{aligned}
&2^p \int_{|f|_Q}^{L/2} p\lambda^{p-1} |\{x \in Q : M_Q^{\mathrm{dy}} f(x) > 2\lambda\}|\, d\lambda \\
&\le 2^{n+p}\gamma \int_{|f|_Q}^{L/2} p\lambda^{p-1} |\{x \in Q : M_Q^{\mathrm{dy}} f(x) > \lambda\}|\, d\lambda \\
&\qquad + 2^p \int_{|f|_Q}^{L/2} p\lambda^{p-1} |\{x \in Q : M_Q^{\sharp,\mathrm{dy}} f(x) > \gamma\lambda\}|\, d\lambda \\
&\le 2^{n+p}\gamma \int_0^{L} p\lambda^{p-1} |\{x \in Q : M_Q^{\mathrm{dy}} f(x) > \lambda\}|\, d\lambda \\
&\qquad + 2^p\gamma^{-p} \int_0^{\infty} p\lambda^{p-1} |\{x \in Q : M_Q^{\sharp,\mathrm{dy}} f(x) > \lambda\}|\, d\lambda.
\end{aligned}
$$

Then, for small $\gamma > 0$,

$$
\begin{aligned}
&(1 - 2^{n+p}\gamma) \int_0^{L} p\lambda^{p-1} |\{x \in Q : M_Q^{\mathrm{dy}} f(x) > \lambda\}|\, d\lambda \\
&\le (2|f|_Q)^p |Q| + 2^p\gamma^{-p} \int_0^{\infty} p\lambda^{p-1} |\{x \in Q : M_Q^{\sharp,\mathrm{dy}} f(x) > \lambda\}|\, d\lambda.
\end{aligned}
$$

Letting $L \to \infty$, we have (3.7). Substitute $f - f_Q$ for f in (3.7). Then

$$
\begin{aligned}
\|f - f_Q\|_{L^p(Q)} &\le \|M_Q^{\mathrm{dy}}(f - f_Q)\|_{L^p(Q)} \\
&\lesssim \|M_Q^{\sharp,\mathrm{dy}} f\|_{L^p(Q)} + |Q|^{1/p} \fint_Q |f - f_Q| \\
&\le \|M_Q^{\sharp,\mathrm{dy}} f\|_{L^p(Q)} + |Q|^{1/p} \inf_{x \in Q} M_Q^{\sharp,\mathrm{dy}} f(x).
\end{aligned}
$$

Since

$$
\begin{aligned}
|Q|^{1/p} \inf_{x \in Q} M_Q^{\sharp,\mathrm{dy}} f(x) &= \left(\int_Q \left[\inf_{x \in Q} M_Q^{\sharp,\mathrm{dy}} f(x) \right]^p dy \right)^{1/p} \\
&\le \|M_Q^{\sharp,\mathrm{dy}} f\|_{L^p(Q)},
\end{aligned}
$$

we have the conclusion. □

Proof of Proposition 3.4. For any ball $B = B(x, r)$, take the cube Q centered at x and with sidelength $2r$. Then $B \subset Q$. By Lemma 3.8 we have

$$
\begin{aligned}
\left(\frac{1}{\varphi(B)} \fint_B |f - f_B|^p \right)^{1/p} &\le \left(\frac{2}{\varphi(B)} \frac{|Q|}{|B|} \fint_Q |f - f_Q|^p \right)^{1/p} \\
&\lesssim \left(\frac{1}{\varphi(B)} \fint_Q (M_Q^{\sharp,\mathrm{dy}} f)^p \right)^{1/p} \\
&\lesssim \|M^\sharp f\|_{L^{(p,\varphi)}(\mathbb{R}^n)}.
\end{aligned}
$$

This shows the conclusion. □

References

[1] R. Arai and E. Nakai, Commutators of Calderón-Zygmund and generalized fractional integral operators on generalized Morrey spaces, Rev. Mat. Complut. published online.

[2] S. Chanillo, A note on commutators, Indiana Univ. Math. J. 31 (1982), no. 1, 7–16.

[3] R. R. Coifman, R. Rochberg and G. Weiss, Factorization theorems for Hardy spaces in several variables, Ann. of Math. (2) 103 (1976), no. 3, 611–635.

[4] G. Di Fazio and M. A. Ragusa, Commutators and Morrey spaces. Boll. Un. Mat. Ital. A (7) 5 (1991), no. 3, 323–332.

[5] S. Janson, Mean oscillation and commutators of singular integral operators. Ark. Mat. 16 (1978), no. 2, 263–270.

[6] Y. Komori-Furuya, Local good-λ estimate for the sharp maximal function and weighted Morrey space. J. Funct. Spaces 2015, Art. ID 651825, 4 pp.

[7] E. Nakai, *Hardy-Littlewood maximal operator, singular integral operators and the Riesz potentials on generalized Morrey spaces*, Math. Nachr. 166 (1994), 95–103.

[8] E. Nakai, *The Campanato, Morrey and Hölder spaces on spaces of homogeneous type*, Studia Math. 176 (2006), no. 1, 1–19.

[9] E. Nakai, *Orlicz-Morrey spaces and the Hardy-Littlewood maximal function*, Studia Math. 188 (2008), no 3, 193–221.

[10] E. Nakai, *Generalized fractional integrals on generalized Morrey spaces*, Math. Nachr. 287 (2014), no. 2-3, 339–351.

[11] E. Nakai and H. Sumitomo, On generalized Riesz potentials and spaces of some smooth functions, Sci. Math. Jpn. 54 (2001), no. 3, 463–472.

[12] E. Nakai and T. Yoneda, Bilinear estimates in dyadic BMO and the Navier-Stokes equations, J. Math. Soc. Japan 64 (2012), no. 2, 399–422.

[13] Y. Tsutsui, Sharp maximal inequalities and its application to some bilinear estimates, J. Fourier Anal. Appl., 17 (2011), 265–289.